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STRESS-STRAIN RELATIONSHIP FOR GRANULAR MATERIALS BASED ON THE HYPOTHESIS OF BEST FIT

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Abstract—Stress strain relationships of a granular system are derived based on the premise that the mean field of displacement is the best fit of actual particle displacements. Based on the best fit hypothesis, two fundamental relationships are derived: (1) the average strain as a function of contact displacements, and (2) the mean field of contact force as a function of stress. These two relationships lead to a stress-strain law without the kinematic constraint of uniform strain. For an assembly with isotropic packing structure, closed-form stress-strain relationships are explicitly derived in which the modulus can be determined from the inter-particle stiffness. The difference between the derived stress-strain relationship and that derived from Voigt hypothesis is discussed. The effect of fabric variation is illustrated in the theory. Discrete element analyses are also performed and the results are used to compare with the estimated moduli from the proposed theory. In the comparison, the solution bounds for granular assembly system are illustrated. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Mechanical behavior of a granular system is of great interest to the area of soil mechanics, powder mechanics, bulk solid mechanics and other related fields. Due to discrete nature of particles, the stress-strain relationship of a granular assembly should be derived with considerations of the inter-particle properties. Along this line, development can be found for regular packing of equal-sized spheres in the earlier work by Deresiewicz (1958) and Duffy and Mindlin (1957), and for random packing of spheres in more recent work by Rothenburg (1988), Walton (1987) and Jenkins (1987). The stress-strain relationship considering the effects of particle rotation was developed by Chang and Liao (1989).

It is noted that the same vehicle used in all the aforementioned studies to achieve a stress-strain relationship is a kinematic hypothesis relating displacements and strains. All the aforementioned studies assumed that the strain is uniform in the packing and that every particle within the assembly displaces in accordance with the applied strain of the assembly. For convenience in the discussion of this paper, we term the uniform strain assumption as the "Voigt hypothesis" and term the displacement field in accordance with the uniform strain as the "mean displacement field".

Although the Voigt hypothesis makes it possible for the derivation of a stress-strain relationship, it restricts the movement of particles and corresponds to an upper-bound solution for the analysis. In this study, we make a more realistic hypothesis for the displacement field in the assembly. We accept the fact that the actual displacement field is not coincide with but fluctuate about the mean displacement field. And we postulate that the mean displacement field is the best fit of the actual displacements. This hypothesis relaxes the kinematic constraint of Voigt assumption.

Based on this hypothesis, two fundamental relationships are derived, namely, (1) the average strain as a function of contact displacements, and (2) the mean field of contact force as a function of stress. Based on this hypothesis, a closed-form expression of the stress-strain relationship can also be derived for an assembly with isotropic packing structure. It can be shown clearly from the analysis that the best fit hypothesis leads to a lower bound solution of the stress-strain relationship. The derived results are compared with that derived from Voigt hypothesis and that calculated from computer simulation for a random packing of round particles. From the comparison, the solution bounds for granular assembly system are illustrated.

2. VOIGT HYPOTHESIS

Based on Voigt hypothesis, two fundamental relationships can be derived, namely (1) the mean field of displacement as a function of strain, and (2) the average stress as a function of contact forces.

2.1. *Mean field of displacements*

According to Voigt hypothesis, the movement of particles in an assembly is in accordance with the mean field of displacement. Under a strain $\dot{\varepsilon}_{ij}$ the mean field of particle displacement can be given by

$$\dot{u}_i^a = \dot{\varepsilon}_{ji} x_j^a \tag{1}$$

where $\dot{u}_i^a$ is the displacement and x_j^a is the position vector of the centroid of particle 'a'. Consider two particles in contact at point 'c' and denote the branch vector l_i^c as the vector connecting the centroids of the two particles. The relative displacement of the two centroids, $\dot{\Delta}_i^c$ can be obtained by:

$$\dot{\Delta}_i^c = \dot{\varepsilon}_{ji} l_j^c. \tag{2}$$

We now consider the simple and plausible assumption commonly used in granular mechanics that particles are relatively rigid and discontinuities are allowed at inter-particle contacts. Then the relative displacements of the two centroids, $\dot{\Delta}_i^c$, represents the dislocation at the inter-particle contact. For convenience, in the following discussion $\dot{\Delta}_i^c$ is denoted as the 'contact displacement'.

2.2. *Average stress from contact forces*

As a consequence of Voigt hypothesis, another important relationship for the average stress can be derived, other than the mean displacement field. Using the principle of potential energy conservation for the assembly, the external work done to the assembly is equal to the internal work done at all inter-particle contacts. It can be written by

$$\sigma_{ij}\dot{\varepsilon}_{ij} = \frac{1}{V}\sum_c f_i^c \dot{\Delta}_i^c \tag{3}$$

where $\sigma_{ij}\dot{\varepsilon}_{ij}$ is the potential energy per volume; σ_{ij} is the stress of the assembly; and f_i^c is the inter-particle contact force vector at contact point 'c'.

Applying Voigt hypothesis to the contact displacement in eqn (3) by substituting eqn (2) into eqn (3), it yields a useful form for the average stress expressed in terms of contact forces, i.e.,

$$\sigma_{ij} = \frac{1}{V}\sum_c l_i^c f_j^c. \tag{4}$$

The Voigt hypothesis in eqn (2) and the average stress in eqn (4) have been widely used as basis for the development of constitutive law in granular mechanics, e.g., Digby (1981), Christoffersen *et al.* (1981), Walton (1987), Jenkins (1988), Chang (1988), etc.

3. BEST FIT HYPOTHESIS

We consider that the actual displacement field is not coincide with but fluctuate about the mean displacement field. The difference between the actual contact displacement $\dot{\Delta}_i^c$ and the mean field displacement is

$$E_i^c = \varepsilon_{ij} l_j^c - \Delta_i^c. \tag{5}$$

We now assume that the mean displacement field best fits of the actual displacements. Using the process of least square, the overall fluctuation is represented by the sum of square of E_i^c for all individual contacts

$$S = \sum_c (E_i^c)^2 = \sum_c (\varepsilon_{ij} l_j^c - \Delta_i^c)^2. \tag{6}$$

The mean displacement field can be obtained from minimizing the value of S. We let the partial derivatives with respect to the strain ε_{ij} be zero, i.e.,

$$\frac{\partial S}{\partial \varepsilon_{mn}} = 0 \tag{7}$$

and we obtain

$$\sum_c (\varepsilon_{ij} l_j^c - \Delta_i^c) \frac{\partial}{\partial \varepsilon_{mn}} (\varepsilon_{ij} l_j^c - \Delta_i^c) = 0. \tag{8}$$

3.1. *Average strain from displacements*

Equation (8) leads to the relationship between the strain and the contact displacements, given by

$$\varepsilon_{ij} = \frac{1}{V} \sum_c \Delta_i^c l_n^c A_{jn} \tag{9}$$

where V is the volume of the representative unit of the assembly and the tensor A_{jn} is defined by

$$A_{jn} = \left[\frac{1}{V} \sum_c l_j^c l_n^c \right]^{-1}. \tag{10}$$

It is noted that the inverse of A_{jn} is a fabric tensor similar to that defined by Satake (1982), Kanatani (1984), Chang and Liao (1994) and Christoffersen *et al.* (1981).

In eqn (9), there are more unknown variables $\dot{\Delta}_i^c$ than the number of equations. Thus, there exists many sets of solutions for $\dot{\Delta}_i^c$ that satisfy eqn (9). Although the mean field of displacement is a set of solution that satisfies eqn (9), it is not necessarily the true solution. In general, eqn (9) alone is not enough information for the determination of the contact displacements based on the strain of the assembly. The nature of eqn (9) is very different from that of eqn (2). However, eqn (9) is a fundamental relationship that can be used to determine the average strain from the contact displacements.

3.2. *Mean field of contact forces*

As a consequence of the best fit hypothesis, another important relationship for the mean contact force field can be derived, other than the average strain. Using the principle of complementary energy conservation for the assembly, it can be written by

$$\dot{\sigma}_{ij}\varepsilon_{ij} = \frac{1}{V}\sum_m \Delta_i^m \dot{f}_i^m \tag{11}$$

where $\dot{\sigma}_{ij}\varepsilon_{ij}$ is the complementary energy per volume; $\dot{\sigma}_{ij}$ is the incremental stress of the assembly; and $\dot{f}_i^c$ is the incremental inter-particle contact force vector at contact point 'c'.

Substituting eqn (9) into eqn (11), it yields

$$\frac{1}{V}\sum_c \Delta_i^c \dot{\sigma}_{ij} l_n^c A_{jn} = \frac{1}{V}\sum_c \Delta_i^c \dot{f}_i^c. \tag{12}$$

Note that in eqn (12), there are more unknown variables $\dot{f}_i^c$ than the number of equations. Thus there exists many sets of solutions for $\dot{f}_i^c$ that satisfy eqn (12). From this equation alone, it is not possible to solve for the true field of contact forces due to the applied stress. However, we can estimate a mean field of contact-forces that satisfy eqn (12), given by

$$\dot{f}_i^c = \dot{\sigma}_{ij} l_n^c A_{jn}. \tag{13}$$

It is interesting to note that equation is identical to the one heuristically assumed earlier by Chang and Liao (1994) and Chang and Gao (1996). Note that the mean field of contact forces in eqn (13) is a set of solution that satisfies the equation of average stress given in eqn (4).

4. PIECE-WIDE FIT HYPOTHESIS

In last section, we obtain the averaged strain of the assembly from the best fit of the displacement field. However, since the strain is known to be non-uniform due to the spacial variation of microstructure, it is desirable to take into account of this effect. To this end, we adopt a more refined piece-wise fit to the displacement field. We first define a 'local region' that consists of a particle and its vicinity. Then a local strain is introduced by fitting the displacements at the local region. The overall strain can then be obtained from the average of local strains.

4.1. *Average strain from displacements*

Consider a local region that consists of particle 'a' and its vicinity. The volume of the local region denoted by V^a is given by

$$V^a = v^a(1+e) \tag{14}$$

where v^a is the volume of the solid particle 'a' and e is the void ratio of assembly defined as the ratio of the volume of voids to the volume of solid particles. The local volume V^a is associated with each particle such that the summation of all local volume equal to the total volume of the representative unit, given by

$$V = \sum_a V^a. \tag{15}$$

For each local region, a local fabric tensor is defined by

$$A^a_{jn} = \left[\frac{1}{2V^a}\sum_{c=1}^{N^a} l^c_j l^c_n\right]^{-1} \tag{16}$$

where N^a is the total number of contact (or the coordination number) of particle 'a'. The volume average of the local fabric tensors is equal to the overall fabric tensor of the representative volume shown in eqn (10):

$$\begin{aligned}[A_{jn}]^{-1} &= \frac{1}{V}\sum_a [A^a_{jn}]^{-1} V^a \\ &= \frac{1}{2V}\sum_a \sum_{c=1}^{N^a} l^c_j l^c_n = \frac{1}{V}\sum_{c=1}^{N} l^c_j l^c_n\end{aligned} \tag{17}$$

where N is the total number of inter-particle contacts in the volume V. Note that in the double summation over contacts of all particles, each contact is counted twice, i.e.,

$$\sum_a \sum_{c=1}^{N^a} = 2\sum_{c=1}^{N}. \tag{18}$$

The process of best fit carried out over a local region is identical to that described in the previous section except that the quantity is now summed over the contacts in the vicinity of particle 'a'.

After minimization through partial derivatives with respect to the strain ε_{ij} and using the definition of local fabric tensor given in eqn (16), we obtain the relationship between the local strain and contact displacements, given by

$$\varepsilon^a_{mj} = \frac{1}{2V^a}\sum_{c=1}^{N^a} \Delta^c_m l^c_n A^a_{jn}. \tag{19}$$

Equation (19) provides the local strain which is an average quantity of the inter-particle contact displacements between particle 'a' and its neighbours.

The overall strain of the representative unit can be defined as the volume average over all local strains, given by

$$\dot{\varepsilon}_{ij} = \frac{1}{V}\sum_a \dot{\varepsilon}^a_{ij} V^a \tag{20}$$

which leads to

$$\dot{\varepsilon}_{ij} = \frac{1}{2V}\sum_a \sum_{c=1}^{N^a} \Delta^c_m l^c_m A^a_{jn}. \tag{21}$$

In the case of regular packing where the local fabric tensor A^a_{jn} is same as the overall fabric tensor of the assembly A_{jn}, then eqn (21) reduces to eqn (9). In general packing conditions, the local fabric varies with space. The strain in eqn (21) is a more refined formulation than eqn (9), that considers the variation of the local fabric.

4.2. *Mean field of contact forces*

Similarly, using the principle of complementary energy conservation, the mean contact-force field for the stress of the particle can be written by

$$\dot{f}_i^c = \dot{\sigma}_{ij}^a l_n^c A_{jn}^a. \tag{22}$$

If uniform stress is assumed and the packing is regular, then eqn (22) is the same as the mean field of contact forces given in eqn (13). Equation (22) shows that, for a random packing, the mean contact force is also a function of the local fabric structure.

5. CONSTITUTIVE LAW FOR GRANULAR ASSEMBLY

5.1. *Based on Voigt hypothesis*

A general relationship between the contact displacement and the contact force at a contact point 'c' can be written as

$$\dot{f}_i^c = K_{ij}^c \dot{\Delta}_j^c \tag{23}$$

where the stiffness tensor K_{ij}^c is a function of elastic properties and the shape of the particle. The stiffness tensor in eqn (23) can be obtained from contact theories for two elastic particles in contact, e.g., Hertz and by Mindlin (1949).

Assuming the contact stiffness are uncoupled between normal and shear directions, the stiffness tensor, K_{ij}^c, can be written as

$$K_{ij}^c = k_n^c n_i^c n_j^c + K_s^c (s_i^c s_j^c + t_i^c t_j^c) \tag{24}$$

where K_n^c and K_s^c are the inter-particle contact stiffness in the normal and tangential directions respectively. The local coordinate system is constructed for each contact point by three orthogonal base unit vectors with the vector **n** normal to the contact area, and the tangent vectors **s** and **t** being chosen arbitrarily.

Therefore, using eqn (23), eqn (2) and eqn (4), and assuming the strain is uniform, then the stress-strain relationship becomes

$$\sigma_{ij} = C_{ijkl} \dot{\varepsilon}_{lk} \tag{25}$$

where

$$C_{ijkl} = \frac{1}{V} \sum_n^e l_i^c K_{jk}^c l_l^c. \tag{26}$$

From the minimum potential energy theorem, eqn (25) gives a upper bound solution of the stress-strain relationship corresponding to the Voigt hypothesis.

5.2. *Based on best fit hypothesis*

The contact law relating the contact displacement and the contact force at a contact point 'c' can be expressed in terms of a flexibility tensor, given as

$$\dot{\Delta}_j^c = S_{ij}^c \dot{f}_i^c. \tag{27}$$

Assuming the contact flexibility are uncoupled between normal and shear direction, the flexibility tensor, S_{ij}^c, can be written as

$$S_{ij}^c = h_n^c n_i^c n_j^c + h_s^c (s_i^c s_j^c + t_i^c t_j^c) \tag{28}$$

where h_n^c and h_s^c are the inverse of K_n^c and K_s^c, respectively, representing the inter-particle contact flexibility in the normal and tangential directions respectively.

Using eqn (27), eqn (13) and eqn (9), the stress-strain relationship is given as follows

$$\dot{\varepsilon}_{ij} = H_{ijkl}\dot{\sigma}_{kl} \tag{29}$$

where

$$H_{ijkl} = \frac{1}{V}\sum_c S^c_{ik} l^c_n l^c_q A_{jn} A_{lq}. \tag{30}$$

$$A^{-1}_{jn} = \frac{1}{V}\sum_c l^c_j l^c_n \tag{31}$$

Equation (30) is the constitutive law derived from the mean force field. However, equilibrium condition is not rigorously guaranteed. Therefore, the stress-strain relationship is only a lower estimate, but not necessarily the lower bound.

5.3. *Based on piece-wise fit hypothesis*

Based on the piece-wise fit assumption, the stress-strain relationship in the form of eqn (29) can also be derived. Using eqn (27), eqn (22) and eqn (19), and assuming the stress to be uniform, i.e., $\sigma_{ij} = \sigma^a_{ij}$, the flexibility tensor in eqn (29) becomes

$$H_{ijkl} = \frac{1}{2V}\sum_a A^a_{mi} \sum_c l^c_m S^c_{jl} l^c_p A^a_{pk}. \tag{32}$$

Equation (32) is also derived from the mean force field. In addition, the stress for all local region are assumed to be equal. Thus, the stress-strain relationship is considered to be a lower bound solution.

5.4. *Representation of packing structure*

It is noted that in the present stress-strain model, information about the complete packing structure is not necessary. For the best fit model, the packing structure is statistically represented by a set of branch vectors. Therefore the packing can be represented by a set of particle pairs oriented in different directions.

For the piece-wise fit model, the packing structure is statistically represented by a set of local fabric tensors. Therefore the packing can be represented by a set of particle groups of different configurations. Each particle group consists of a particle and its neighbouring particles.

Since the complete packing structure in detail is not required, it greatly reduces the computing effort compared to other types of analysis, e.g., discrete element method. Under some idealized conditions, closed-form expressions of the stress-strain relationship can be obtained such as the ones derived in the next section.

6. CONSTITUTIVE LAW FOR ASSEMBLY WITH ISOTROPIC PACKING STRUCTURE

In this section, we derive closed-form stress-strain relationships for a granular material with isotropic packing structure under elastic condition. The packing of the assembly is composed of a large number of round particles. All particles are of the same size and same material properties.

For a suitably large representative volume with a large number of particles, the summation can be expressed in an integral form. Let f be a quantity dependent on the orientation of contact, the summation of such a function over all contacts can be written as

$$\sum_{c=1}^{N} f(\gamma,\beta) = \frac{N}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{33}$$

where γ and β are respectively the angles of rotation and declination in a spherical coordinate system. Using eqn (33), the derived constitutive relations in summation form can be written in an integral form. For the example of packings with equal sized particles, the stiffness tensor based on Voigt hypothesis becomes:

$$C_{ijkl} = \frac{Nr^2}{\pi V}\int_0^{2\pi}\int_0^{\pi} n_i(\gamma,\beta)K_{jk}(\gamma,\beta)n_l(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{36}$$

and the flexibility tensor based on best fit hypothesis becomes:

$$H_{ijkl} = \frac{9V}{16\pi Nr^2}\int_0^{2\pi}\int_0^{\pi} n_i(\gamma,\beta)S_{jk}(\gamma,\beta)n_l(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta \tag{37}$$

where r is the radius of particle, N is the total number of inter-particle contacts in the representative volume V.

For the stress-strain relationship based on the hypothesis of piece-wise fit, the flexibility tensor can not be expressed as a simple integration form because it involves a double summation, not only over the contacts but also over the particles.

Proceeding from eqn (36) and (37), we now seek to derive the closed-form expressions for the stress-strain relationships based on Voigt hypothesis and on the best fit hypothesis. After carrying out the integration, the resulting stress-strain relationship becomes

$$\sigma_{ij} = \frac{Nr^2}{15V}(2K_n+3K_s)\left[\varepsilon_{ij}+\frac{K_n-K_s}{2K_n+3K_s}\varepsilon_{kk}\delta_{ij}\right] \tag{38}$$

for the Voigt model, and becomes

$$\varepsilon_{ij} = \frac{3V}{5Nr^2K_n^2}[(2K_s+3K_n)\sigma_{ij}+(K_s-K_n)\sigma_{kk}\delta_{ij}] \tag{39}$$

for the best-fit model. For the purpose of comparing this equation with eqn (38), we inverse eqn (39) to the following form:

$$\sigma_{ij} = \frac{Nr^2K_n}{3V}\frac{5K_s}{3K_n+2K_s}\left[\varepsilon_{ij}+\frac{K_n-K_s}{5K_s}\varepsilon_{kk}\delta_{ij}\right] \tag{40}$$

In the theory of solid mechanics, the usual stress-strain relationship for an isotropic elastic material is given by

$$\sigma_{ij} = \frac{E}{1+v}\left[\varepsilon_{ij}+\frac{v}{1-2v}\varepsilon_{kk}\delta_{ij}\right] \tag{41}$$

where E and v are, respectively, Young's modulus and Poisson's ratio. By comparing eqn (41) with eqns (38) and (40), the equivalent Young's modulus E, the equivalent Poisson's ratio v and the equivalent bulk modulus B for granular materials are obtained and listed below:

(1) Based on Voigt hypothesis

For assemblies with spherical particles

$$E = \frac{4Nr^2K_n}{3V}\left[\frac{2K_n+3K_s}{4K_n+K_s}\right]; \quad \nu = \frac{K_n-K_s}{4K_n+K_s}$$

$$B = \frac{4Nr^2}{9V}K_n. \tag{42}$$

For assemblies with disk particles

$$E = \frac{4Nr^2K_n}{V}\left[\frac{K_n+K_s}{3K_n+K_s}\right]; \quad \nu = \frac{K_n-K_s}{3K_n+K_s}$$

$$B = \frac{Nr^2}{V}K_n. \tag{43}$$

(2) Based on the best fit hypothesis
For assemblies with spherical particles

$$E = \frac{20Nr^2K_n}{3V}\left[\frac{K_s}{2K_n+3K_s}\right]; \quad \nu = \frac{K_n-K_s}{2K_n+3K_s}$$

$$B = \frac{4Nr^2}{9V}K_n. \tag{44}$$

For assemblies with disk particles

$$E = \frac{8Nr^2K_n}{V}\left[\frac{K_s}{K_n+3K_s}\right]; \quad \nu = \frac{K_n-K_s}{K_n+3K_s}$$

$$B = \frac{Nr^2}{V}K_n. \tag{45}$$

According to eqns (42)–(44), the derived bulk moduli based on Voigt hypothesis is identical to that based the best-fit hypothesis. However, the derived Young's modulus and Poisson's ratio are different based on the two different hypotheses. Using eqns (42) and (44) for packings of spheres, Young's modulus is plotted in Fig. 1 and Poisson's ratio is plotted in Fig. 2 against the ratio of K_s and K_n. Under the condition of $K_s = K_n$, the moduli and Poisson's ratios are same for both Voigt and the best fit models. When K_s is much smaller than K_n, the differences become large. When the tangential stiffness K_s approaches to zero, in the best fit model, Poisson's ratio approaches to 0.5 and Young's modulus approaches to zero. Physically, it implies that a granular assembly with perfectly smooth particles (i.e., zero inter-particle tangential stiffness) has no shear modulus but retains bulk modulus. The granular assembly thus can resist only the isotropic bulk stress but not the deviatoric stress; it behaves like fluid. However, in Voigt model, the granular assembly with zero inter-particle tangential stiffness still has moderate shear modulus and behaves like solid.

The two types of hypotheses have distinctive differences in the predicted behavior for a granular assembly. In order to have a clearer perspective of the predicted behavior in relation to the true behavior, we examine the limitations of the two hypothesis. The limitation of Voigt hypothesis is its assumption of uniform strain for the entire region of the granular assembly. This limitation is not applied to the best-fit hypothesis in which the assumed displacement field is a least-square fit of the true displacements of particles.

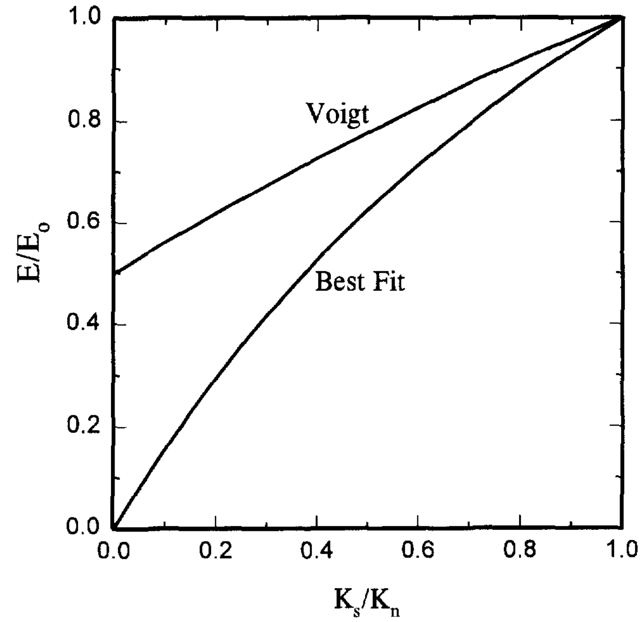

Fig. 1. Young's modulus for a packing of spheres based on Voigt hypothesis and the best fit hypothesis.

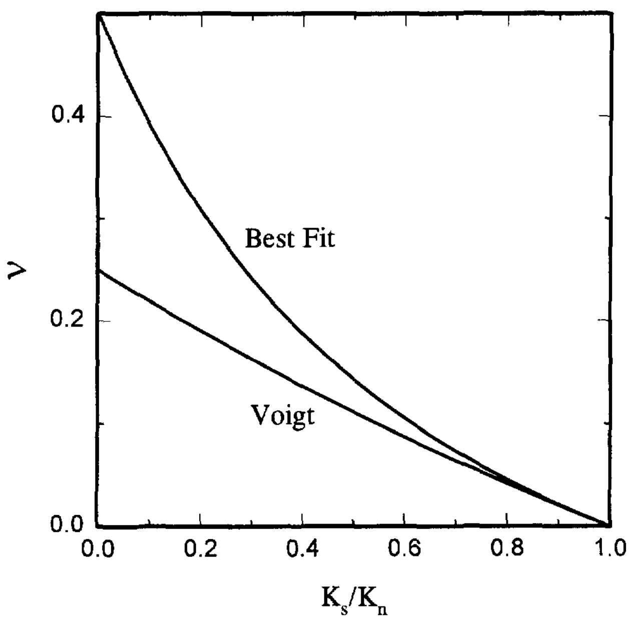

Fig. 2. Poisson's ratio for a packing of spheres based on Voigt hypothesis and the best fit hypothesis.

However, the best-fit displacement field retorts to another type of limitation with which the displacement field does not necessarily satisfy the kinematic compatibility between 'local regions'. The condition of compatibility, on the other hand, is guaranteed by Voigt hypothesis of uniform strain.

Uniform strain is an extremely restrictive assumption on the displacement field so that we expect the Voigt hypothesis leads to a prediction of stiffer behavior. On the contrary, the best-fit hypothesis allows freedom for the displacement field to fit the true displacements of particles without the constraints of compatibility between local regions. Therefore the predicted behavior is expected to be softer than the true behavior.

Using eqns (42) and (44) for packings of disks, Young's modulus and Poission's ratio against the ratio of K_s and K_n are plotted in Figs 3 and 4. The trends are similar to that of the packing of spheres.

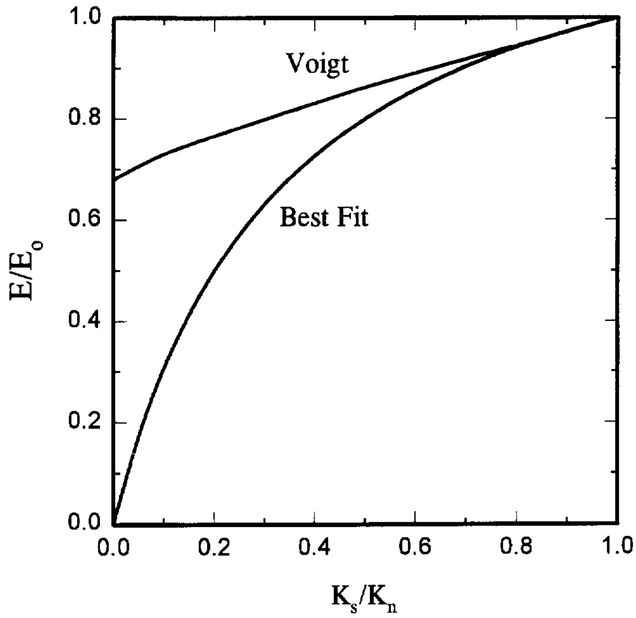

Fig. 3. Young's modulus for a packing of disks based on Voigt hypothesis and the best fit hypothesis.

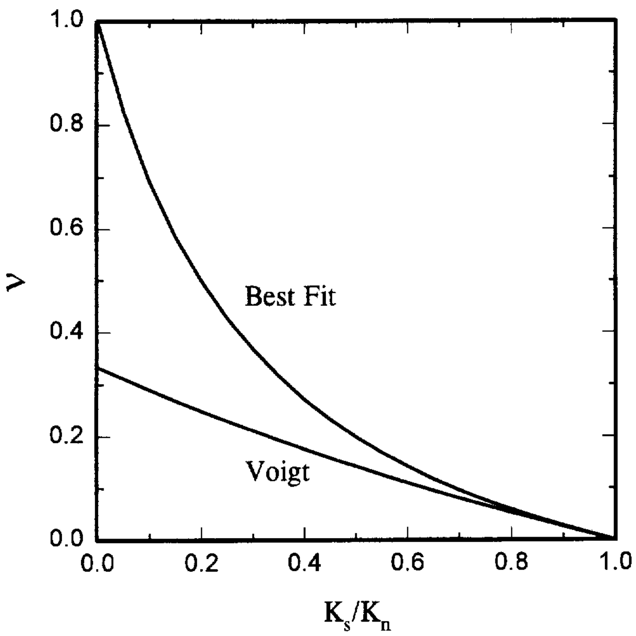

Fig. 4. Poisson's ratio for a packing of disks based on Voigt hypothesis and the best fit hypothesis.

7. NUMERICAL EXAMPLES

In the previous section, we have derived closed-form stress-strain relationships of an isotropic granular assembly using Voigt model and the best fit model. We now use numerical calculation to obtain the elastic modulus using the piece-wise model. In this analysis, information about local fabric structure is required. We proceed this numerical analysis with two different ways of determining the local fabric structure. In the first example, the local fabric structures are generated from random generation. In the second example, we create a complete packing structure in the discrete element analysis. Then the local fabric structures are sampled from the packing.

7.1. *Random generation of local fabric structure*

The configurations of a particle group, consisting of a particle and its neighbouring particles, is randomly generated. We generate many particle groups to statistically represent the isotropic packing structure of a granular assembly. Here we use 800 particle groups to represent a random packing structure.

In random packing, the local fabric are inevitably anisotropic. However, in the random generation process, the axes of local fabric anisotropy are equally possible to occur in all orientations. The 800 groups are used to compute the local fabric tensor in eqn (16). The overall fabric tensor for the packing, averaged from the 800 local fabric tensors according to eqn (17), reasonably retains the overall isotropy. Then the overall moduli of the packing is determined based on eqn (32).

To study the effect of coordination number on local fabric and the resulting modulus, we let all 800 groups be made of same number of particles. Four different packing structures are considered: i.e., the number of neighbouring particles $N^a = 3, 4, 5$ and 6. The elastic modulus estimated from the piece-wise fit hypothesis is plotted in Fig. 5 compared with that estimated from Voigt hypothesis.

For a particle with less neighbours, there is more freedom for the neighbours to form different configurations, and it is possible to form highly anisotropic configurations in the local region. On the other hand, for a particle with more neighbours, it is less possible to have a highly anisotropic configuration. For example in the packing with equal sized disks, a particle with six neighbours is the densest packing structure and can only have a single possible configuration. Under this condition, the local fabric tensor is found to be same for all particle groups (i.e., $A^a_{ij} = A_{ij}$). Thus the predicted modulus with $N^a = 6$ obtained from the piece-wise fit model, as shown in Fig. 5, is identical to that derived from the best-fit model.

Compare to Fig. 4 and Fig. 5, a significant difference can be observed between the best fit model and the piece-wise fit model, particularly when $K_n = K_s$. This deviation is caused from the heterogeneity of the granular packing structure that is considered in the piece-wise fit model. Figure 5 shows that modulus decreases with the coordination number because a lower N^a indicates a more variation of the local fabric structure. Thus the packing

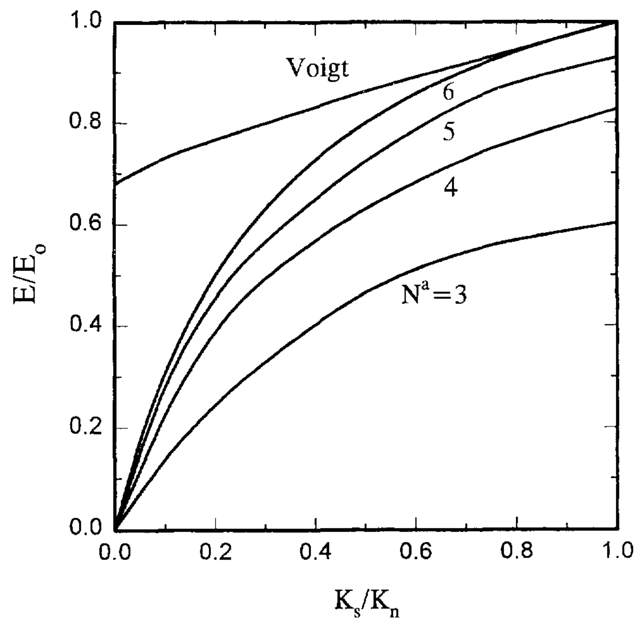

Fig. 5. Young's modulus obtained from the piece-wise fit hypothesis for packings of disks with different coordination number.

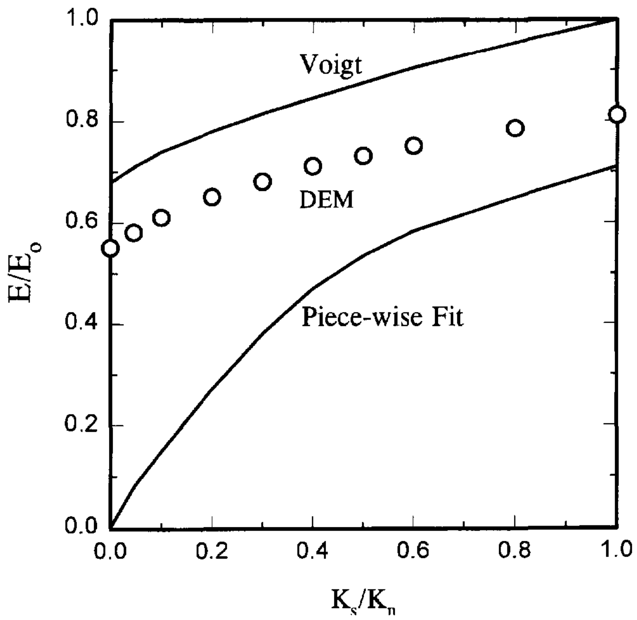

Fig. 6. Comparison of Young's modulus obtained from (1) Voigt hypothesis, (2) piece-wise fit hypothesis, and (3) discrete element method.

with $N^a = 3$ has the lowest estimate of modulus. This is in agreement with the anisotropic and heterogeneous nature of loose microstructure described previously.

7.2. *Comparison with DEM results*

Several random packings consisted of approximately 800 circular disks with equal size are used to evaluate the moduli using the discrete element method (DEM). The elastic moduli derived from the DEM method are then compared with the moduli derived from Voigt model and the piece-wise fit model.

For the piece-wise fit model, the packing structure is represented by a set of particle groups of different configurations. Particle groups are obtained by sampling directly from the complete packing structure.

The results for all packings show similar behavior trend. The results of a typical packing are plotted in Fig. 6. The packing consists of 897 with a periodic packing structure. Total number of contacts is 2306 and the average coordination number 5.142.

The modulus obtained from DEM is plotted in Fig. 6, compared with the modulus estimated from Voigt and piece-wise hypotheses. It shows that these two solutions can be served as the upper and lower bound of the modulus.

8. CONCLUSION

Voigt hypothesis has been widely used in granular mechanics for estimating the stress-strain behavior of randomly packed particles. In this study, we relax the kinematic constrains of Voigt hypothesis and use a best fit hypothesis for the displacement field in the assembly. Based on the best fit hypothesis, two fundamental relationships are derived: (1) the average strain from contact displacements and (2) the mean field of contact forces from stress. These two relationships lead to a stress-strain relationship that is different from the usual ones derived from Voigt hypothesis.

The following conclusions can be drawn from this paper:

(1) The stress-strain relationship derived from the best fit hypothesis show very different behavior from that derived from the Voigt hypothesis. For example in the cases of particles with zero tangential stiffness, the assembly has a fluid-like instead of a solid-like behavior.

(2) The piece-wise fit model show the effect of local fabric variation as a result of the structural heterogeneity of the granular packing. The variation of local fabric in a loose packing is a significant factor influencing the modulus of the packing.

(3) The piece-wise fit model is derived based on a mean field of contact forces with the assumption of uniform stress for all local regions in the packing. Thus it is a lower bound solution. Together with the upper solution from Voigt hypothesis, it provides useful estimates for the range of the true solution.

REFERENCES

Bathurst, R. J. and Rothenburg, L. (1988) Micromechanical aspects of isotropic granular assemblies with linear contact interactions. *Journal of Applied Mechanics, ASME* **55**(1), 17–23.

Chang, C. S. (1987) Micromechanical modelling of constitutive relations for granular material. In *Micromechanics of Granular Materials*, eds M. Satake and J. T. Jenkins. Elsevier, Amsterdam, pp. 271–279.

Chang, C. S. and Misra, A. (1989) Theoretical and experimental study of regular packing of granules. *Journal of Engineering Mechanics* **4**, x–y.

Chang, C. S. and Liao, C. (1989) Constitutive relations for particulate medium with the effect of particle rotation. *International Journal of Solids and Structures* **26**, 437–453.

Liao, C. L., Chang, T. C. and Chan, H. S. (1991) The investigation of multi-axial stress-strain relationship for concrete material in micro-structural model. In *1st Structural Engineering Conference*, R.O.C., pp. 67–77.

Chang, C. S. and Gao, J., (1996) Kinematic and static hypotheses for constitutive modelling of granulates considering particle rotation. *Acta Mechanica* **115**, 213–239.

Chang, C. S. and Liao, C. L. (1994) Estimates of elastic modulus for media of randomly packed granules. *Applied Mechanical Review* **47**, S197–S206.

Christoffersen, J., Mehrabadi, H. M. and Nemat-Nasser, S. (1981) A micromechanical description of granular material behavior. *Journal of Applied Mechanics, ASME* **48**(2), 339–344.

Cundall, P. and Strack, O. D. L., (1979) A discrete numerical model for granular assemblies. *Geotechnique* **29**(1), 47–65.

Deresiewicz, H. (1958) Stress-strain relations for a simple model of a granular medium. *Journal of Applied Mechanics, Transactions ASME* **25**(3), 402–406.

Duffy, J., (1959) A differential stress-strain relation for the hexagonal close packed array. *Journal of Applied Mechanics, Transactions ASME* **7**, 88–94.

Duffy, J. and Mindlin, R. D. (1957) Stress-strain relations and vibrations of a granular media. *Journal of Applied Mechanics, Transactions ASME* **X**, 585–593.

Jenkins, J. T. (1987) Volume change in small strain axisymmetric deformations of a granular material. In *Micromechanics of Granular Materials*, eds M. Satake and J. T. Jenkins. Elsevier, Amsterdam, pp. 245–252.

Kanatani K. (1984) Distribution of directional data and fabric tensors. *International Journal of Engineering Science* **22**(2), 149–164.

Kishino, Y. (1987) Disc model analysis of granular media. In *Micromechanics of Granular Materials*, eds M. Satake and J. T. Jenkins. Elsevier, Amsterdam, pp. 143–152.

Mindlin, R. D. (1949) Compliance of elastic solids in contact. *Journal of Applied Mechanics, ASME* **x**, 259–268.

Rothenburg, L. and Selvadurai, A. P. S. (1981) Micromechanical definition of the Cauchy stress tensor for particulate media. In *Mechanics of Structured Media*, ed. A. P. S. Selvadurai. Elsevier, Oxford, pp. 469–486.

Satake, M. (1982) Fabric tensor in granular materials. In *Proceedings, IUTAM symposium on Deformation and Failure of Granular Materials*, eds P. A. Vermeer and H. J. Luger. A. A. Balkema, Delft, pp. 63–68.

Serrano, A. A. and Rodrigues-Ortiz, J. M. (1973) A contribution to the mechanics of heterogeneous granular media. In *Proceedings of the Symposium on Plasticity and Soil Mechanics*, Cambridge, pp. 215–227.

Truesdell, C. and Toupin, R. A. (1960) The classical field theories. *Handbuch der Physik*, ed. S. Flugge, Vol. III/1. Springer Verlag, Berlin.

Walton, K. (1987) The effective elastic moduli of a random packing of spheres. *Journal of Mechanics and Physics of Solids* **35**, 13–226.